On the L^1 norm of exponential sums

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23 May 2023

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Littlewood conjecture

In 1948, Littlewood tempted to ask the following question : Is it true that, when $n_1 < \cdots < n_N$ are integers

$$\int_{-1/2}^{1/2} \left| \sum_{k=1}^{N} e^{2i\pi n_k t} \right| \mathrm{d}t \geq \int_{-1/2}^{1/2} \left| \sum_{k=1}^{N} e^{2i\pi kt} \right| \mathrm{d}t.$$

Littlewood did not explicitly ask this question and made a safer guess. He conjectured that, when $n_1 < n_2 < \cdots < n_N$ are integers, there exists a universal constant C such that

$$L_N := \inf_{n_1 < n_2 < \cdots < n_N} \int_{-1/2}^{1/2} \left| \sum_{k=1}^{N} e^{2i\pi n_k t} \right| \mathrm{d}t \geq C \log N.$$

Partial results

The first non-trivial estimate was obtained by Cohen who proved that

$$L_N \geq C(\ln N/\ln\ln N)^{1/8}.$$

Subsequent improvements are due to Pichorides who proved that

$$L_N \geq C\ln N/(\ln\ln N)^2.$$

Theorem (MPS Theorem 1981)

For $n_1 < n_2 < \cdots < n_N$ *integers and* $a_1, \ldots, a_N$ *complex numbers,*

$$\int_{-1/2}^{1/2} \left| \sum_{k=1}^{N} a_k e^{2i\pi n_k t} \right| \mathrm{d}t \geq \frac{1}{30} \sum_{k=1}^{N} \frac{|a_k|}{k}.$$

On The Constant In The Littlewood Problem

Theorem (Stegemann & Yabuta Theorem 1982)

If $n_1 < n_2 < \cdots < n_N$ integers and $a_1, \ldots, a_N$ complex numbers all of modulus larger than 1 then

$$\int_{-1/2}^{1/2} \left| \sum_{k=1}^{N} a_k e^{2i\pi n_k t} \right| \mathrm{d}t \geq \frac{4}{\pi^3} \ln N.$$

Theorem (H&L Theorem 1992)

For $\lambda_1 < \lambda_2 < \cdots < \lambda_N$ real numbers and $a_1, \ldots, a_N$ complex numbers,

$$\lim_{T \to +\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left| \sum_{k=1}^{N} a_k e^{2i\pi\lambda_k t} \right| \mathrm{d}t \geq \frac{1}{30} \sum_{k=1}^{N} \frac{|a_k|}{k}.$$

Theorem (Nazarov Theorem 1996)

For $T > 1$, there exists a constant C_T such that, for $\lambda_1 < \lambda_2 < \cdots < \lambda_N$ real numbers verifying $|\lambda_k - \lambda_\ell| \geq |k - \ell|$ and $a_1, \ldots, a_N$ complex numbers,

$$\int_{-T/2}^{T/2} \left| \sum_{k=1}^{N} a_k e^{2i\pi\lambda_k t} \right| \mathrm{d}t \geq C_T \sum_{k=1}^{N} \frac{|a_k|}{k}.$$

Theorem (P.Jaming, K.Kellay and C.Saba 2023)

For $\lambda_1 < \lambda_2 < \cdots < \lambda_N$ real numbers and $a_1, \ldots, a_N$ complex numbers :

1. $$\lim_{T\to+\infty} \frac{1}{T}\int_{-T/2}^{T/2} \left|\sum_{k=1}^{N} a_k e^{2i\pi\lambda_k t}\right| \mathrm{d}t \geq \frac{1}{26}\sum_{k=1}^{N}\frac{|a_k|}{k+1}.$$

2. *If further $a_1, \ldots, a_N$ all have modulus larger than 1, then*
$$\lim_{T\to+\infty} \frac{1}{T}\int_{-T/2}^{T/2} \left|\sum_{k=1}^{N} a_k e^{2i\pi\lambda_k t}\right| \mathrm{d}t \geq \frac{4}{\pi^3}\ln N.$$

3. *If $|\lambda_k - \lambda_\ell| \geq |k-\ell|$ for $k, \ell = 1, \ldots, N$, then for $T \geq 72$;*
$$\frac{1}{T}\int_{-T/2}^{T/2} \left|\sum_{k=1}^{N} a_k e^{2i\pi\lambda_k t}\right| \mathrm{d}t \geq \frac{1}{122}\sum_{k=1}^{N}\frac{|a_k|}{k+1}.$$

1. Validity of Nazarov theorem in the case $T = 1$;
2. Generalisation to sets with multidimensional structure ;
3. Quantitative version of Nazarov Theorem for T small.

References

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Thank you