The Littlewood Problem And Non-Harmonic Fourier Series.

Chadi SABA joint work with P. JAMING and K. KELLAY.

We give a quantitative estimate of L^1 norms of non-harmonic trigonometric polynomials. This extends the result of Konyagin and Mc Gehee, Pigno, Smith.

Institut de Mathématiques de Bordeaux

1. Littlewood Conjecture

In 1948 Littlewood conjectured that, when $n_1 < n_2 < \cdots < n_N$ are integers, there exists a universal constant C such that

$$L_N := \inf_{n_1<n_2<\cdots<n_N} \int_{-1/2}^{1/2} \left| \sum_{k=1}^{N} e^{2i\pi n_k t} \right| \mathrm{d}t \geq C \log N.$$

The first non-trivial estimate was obtained by Cohen who proved that $L_N \geq C(\ln N/\ln\ln N)^{1/8}$ for $N \geq 4$. Subsequent improvements are due to Pichorides who proved that $L_N \geq C \ln N/(\ln\ln N)^2$. In 1981, Littlewood's conjecture was proved by Konyagin and Mc Gehee, Pigno, Smith. Improvements on the constant were made by Stegeman and Yabuta and a generalisation to the real case is due to Nazarov, Hudson and Leckband.

2. MPS Theorem

For $n_1 < n_2 < \cdots < n_N$ integers and $a_1, \ldots, a_N$ complex numbers,

$$\int_{-1/2}^{1/2} \left| \sum_{k=1}^{N} a_k e^{2i\pi n_k t} \right| \mathrm{d}t \geq \frac{1}{30} \sum_{k=1}^{N} \frac{|a_k|}{k}.$$

The particular case $(a_k)_k = 1$ leads to the solution of the Littlewood problem.

3.Stegeman & Yabuta Theorem

If $n_1 < n_2 < \cdots < n_N$ integers and $a_1, \ldots, a_N$ complex numbers all of modulus larger than 1 then

$$\int_{-1/2}^{1/2} \left| \sum_{k=1}^{N} a_k e^{2i\pi n_k t} \right| \mathrm{d}t \geq \frac{4}{\pi^3} \ln N.$$

4. Hudson & Leckband Theorem

Hudson and Leckband extended previous result to non-integer frequencies by using a perturbation argument.

Theorem. *For $\lambda_1 < \lambda_2 < \cdots < \lambda_N$ real numbers and $a_1, \ldots, a_N$ complex numbers,*

$$\lim_{T\to+\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left| \sum_{k=1}^{N} a_k e^{2i\pi\lambda_k t} \right| \mathrm{d}t \geq \frac{1}{30} \sum_{k=1}^{N} \frac{|a_k|}{k}.$$

5. Nazarov Theorem

Nazarov showed that such a result holds not only when $T \to +\infty$ but as soon as $T > 1$:

Theorem. *For* $\mathbf{T > 1}$, *there exists a constant C_T such that, for $\lambda_1 < \lambda_2 < \cdots < \lambda_N$ real numbers verifying $|\lambda_k - \lambda_\ell| \geq |k - \ell|$ and $a_1, \ldots, a_N$ complex numbers,*

$$\int_{-T/2}^{T/2} \left| \sum_{k=1}^{N} a_k e^{2i\pi\lambda_k t} \right| \mathrm{d}t \geq C_T \sum_{k=1}^{N} \frac{|a_k|}{k}.$$

$C_T = ??$

6. Quantitative Extension Of Nazarov Theorem

Let $\lambda_1 < \lambda_2 < \cdots < \lambda_N$ be N distinct real numbers and $a_1, \ldots, a_N$ be complex numbers. Then

172 We have

$$\lim_{T\to+\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left| \sum_{k=1}^{N} a_k e^{2i\pi\lambda_k t} \right| \mathrm{d}t \geq \frac{1}{26} \sum_{k=1}^{N} \frac{|a_k|}{k+1}.$$

173 If further $a_1, \ldots, a_N$ all have modulus larger than 1, $|a_k| \geq 1$ then

$$\lim_{T\to+\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left| \sum_{k=1}^{N} a_k e^{2i\pi\lambda_k t} \right| \mathrm{d}t \geq \frac{4}{\pi^3} \ln N.$$

174 Assume further that for $k, \ell = 1, \ldots, N$, $|\lambda_k - \lambda_\ell| \geq |k - \ell|$, then for $T \geq 72$ we have

$$\frac{1}{T} \int_{-T/2}^{T/2} \left| \sum_{k=1}^{N} a_k e^{2i\pi\lambda_k t} \right| \mathrm{d}t \geq \frac{1}{122} \sum_{k=1}^{N} \frac{|a_k|}{k+1}.$$

7. Strategy Of The Proof

Let

$$\phi(t) = \sum_{k=1}^{N} a_k e^{2\pi i\lambda_k t} \quad \text{and} \quad S = \sum_{k=1}^{N} \frac{|a_k|}{k}.$$

We then write $|a_k| = a_k u_k$ with u_k complex numbers of modulus 1 and define $U(t) = \sum_{k=1}^{N} \frac{u_k}{k} e^{2\pi i\lambda_k t}$.

By orthogonality, $S = \lim_{T\to+\infty} \frac{1}{T} \int_{-T/2}^{T/2} \phi(t)U(t)\,\mathrm{d}t$. The second step will consist in correcting U into V in such a way that $\|V\|_\infty \leq A$ where A is a constant then we multiply by a_k and sum over k to get

$$\lim_{T\to+\infty} \frac{1}{T} \left| \int_{-T/2}^{T/2} \big(U(t) - V(t)\big)\phi(t)\,\mathrm{d}t \right| \leq \alpha S.$$

Then, as

$$S = \lim_{T\to+\infty} \frac{1}{T} \left(\int_{-T/2}^{T/2} \phi(t)V(t)\,\mathrm{d}t + \int_{-T/2}^{T/2} \phi(t)\big(U(t) - V(t)\big)\,\mathrm{d}t \right),$$

we obtain

$$S \leq \|V\|_\infty \lim_{T\to+\infty} \frac{1}{T} \int_{-T/2}^{T/2} |\phi(t)|\,\mathrm{d}t + \alpha S,$$

that is

$$S \leq \frac{A}{1-\alpha} \lim_{T\to+\infty} \frac{1}{T} \int_{-T/2}^{T/2} |\phi(t)|\,\mathrm{d}t$$

as desired.

8. Perspectives

- Validity of Nazarov theorem in the case $T = 1$.
- Generalisation to the multidimensional case ($n_k \in \mathbb{Z}^r$, $r > 1$).
- Quantitative version of Nazarov Theorem For T small enough .

9. References

[1] Godfrey Harold Hardy and John Edensor Littlewood. A new proof of a theorem on rearrangements. *Journal of the London Mathematical Society*, 1(3):163–168, 1948.

[2] O Carruth McGehee, Louis Pigno, and Brent Smith. Hardy's inequality and the l1 norm of exponential sums. *Annals of Mathematics*, pages 613–618, 1981.

[3] Fedor L'vovich Nazarov. On a proof of the littlewood conjecture by mcgehee, pigno and smith. *Algebra i Analiz*, 7(2):106–120, 1995.