Quantitative version of Nazarov's theorem

Chadi SABA

University of Bordeaux

25 Octobre 2022

Contents

Littlewood Conjecture

In 1948 appears at the end of an article signed by Hardy and Littlewood the following question known as the Littlewood conjecture : let $\lambda_1 < \cdots < \lambda_N$ a sequence of N distinct integers. Let

$$\phi(x) = \sum_{k=1}^{N} e^{2i\pi\lambda_k x} \quad \text{and} \quad \|\phi\|_1 = \int_{-\frac{1}{2}}^{\frac{1}{2}} |\phi(x)| \,\mathrm{d}x.$$

Then can we (always) find a positive constant c such that

$$\|\phi\|_1 \geqslant c \ln N.$$

Mc-Gehee, Pigno and Smith Theorem (1981)

The solution was found almost simultaneously and with different methods, by Konyagin, then by McGehee, Pigno and Smith.

Mc-Gehee, Pigno and Smith Theorem (1981)

Theorem (MPS solution of the Littlewood conjecture)

There exists $A \geqslant 1$ such that, for all finite sequence $\lambda_1 < \cdots < \lambda_N$ of integers and all sequence $a_1, \ldots, a_N$ of complex numbers we have

$$\sum_{k=1}^{N} \frac{|a_k|}{k} \leqslant A \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \sum_{k=1}^{N} a_k e^{2i\pi\lambda_k x} \right| dx.$$

On The Constant In The Littlewood Problem

Several mathematicians have already worked on the subject of quantifying the constant :

On The Constant In The Littlewood Problem

- McGehee, Pigno and smith proved that we can take $c = \dfrac{1}{128}$.

On The Constant In The Littlewood Problem

- Stegeman proved that $c \geqslant \frac{4}{\pi^3}$.

On The Constant In The Littlewood Problem

- In the case $\lambda_k = k$, using well-known properties of Dirichlet kernel defined as follow

$$D_N(x) = \sum_{k=1}^{N} e^{ikx},$$

we obtain $c \geqslant \frac{1}{\pi}$.

Nazarov extended the result of MPS the case of real frequencies $\lambda_1 < \cdots < \lambda_N$ verifying $\lambda_{k+1} - \lambda_k \geqslant 1$.

Theorem (Nazarov 1995)

Let $T > 1$, then there exists a strictly positive constant A_T such that, for all real sequence $\lambda_1 < \cdots < \lambda_N$ verifying $\lambda_{k+1} - \lambda_k \geqslant 1$ and all sequence $a_1, \ldots, a_N$ of complex numbers we have

$$\sum_{k=1}^{N} \frac{|a_k|}{k} \leqslant A_T \int_{-\frac{T}{2}}^{\frac{T}{2}} \left| \sum_{k=1}^{N} a_k e^{2i\pi\lambda_k x} \right| \mathrm{d}x.$$

Theorem (Nazarov 1995)

Let $T > 1$, then there exists a strictly positive constant A_T such that, for all real sequence $\lambda_1 < \cdots < \lambda_N$ verifying $\lambda_{k+1} - \lambda_k \geqslant 1$ and all sequence $a_1, \ldots, a_N$ of complex numbers we have

$$\sum_{k=1}^{N} \frac{|a_k|}{k} \leqslant A_T \int_{-\frac{T}{2}}^{\frac{T}{2}} \left| \sum_{k=1}^{N} a_k e^{2i\pi\lambda_k x} \right| \mathrm{d}x.$$

Problem : Explicit form of the constant A_T ? ? ?

Quantitative Version Of Nazarov Theorem

Theorem

Let $T \geqslant 2$. Then there exists a strictly positive constant A (independant of T) such that, for all real sequence $\lambda_1 < \cdots < \lambda_N$ verifying $\lambda_{k+1} - \lambda_k \geqslant 1$ and all sequence $a_1, \ldots, a_N$ of complex numbers we have

$$\sum_{k=1}^{N} \frac{|a_k|}{k} \leqslant \frac{A}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \left| \sum_{k=1}^{N} a_k e^{2i\pi\lambda_k x} \right| \mathrm{d}x.$$

First step

Let N_T a strictly positive integer. As $k + N_T \leqslant k(1 + N_T)$, for $k \geqslant 1$, then

$$\sum_{k=1}^{N} \frac{|a_k|}{k} \leqslant (1 + N_T) \sum_{k=1}^{N} \frac{|a_k|}{k + N_T},$$

and we can prove that $\displaystyle\sum_{k=1}^{N} \frac{|a_k|}{k + N_T} \leqslant B_T$ to get

$$A_T = (1 + N_T) B_T$$

We write $|a_k| = a_k u_k$ where $|u_k| = 1$. Let $I_j = [\![2^j, 2^{j+1} [\![$,

$$f_j(x) = \sum_{r+N_T \in I_j} \frac{u_r}{r+N_T} e^{-2i\pi\lambda_r x} \text{ and } L_0(x) = \sum_{j=1}^{N} f_j(x)$$

We write $|a_k| = a_k u_k$ where $|u_k| = 1$. Let $I_j = [\![2^j, 2^{j+1}[\![$,

$$f_j(x) = \sum_{r+N_T \in I_j} \frac{u_r}{r+N_T} e^{-2i\pi\lambda_r x} \text{ and } L_0(x) = \sum_{j=1}^{N} f_j(x)$$

Lemma (1)

There exist $\alpha \in]0,1[$ such that, for $1 \leqslant k \leqslant N$

$$\left| \int_{-\frac{T}{2}}^{\frac{T}{2}} L_0(x) e^{2i\pi\lambda_k x} \varphi(x)\,\mathrm{d}x - \frac{u_k}{k+N_T} \right| \leqslant \frac{1-\alpha}{k+N_T} \qquad (1)$$

Let ε a small real number, h_j a function verifying $\Re(h_j) = |f_j|$ and $\varphi(x)$ as previously defined. We introduce

$$L_1(x) = \sum_{j=1}^{N} f_j(x) e^{-\varepsilon(h_{j+1}(x)+\ldots+h_N(x))}.$$

Let ε a small real number, h_j a function verifying $\Re(h_j) = |f_j|$ and $\varphi(x)$ as previously defined. We introduce

$$L_1(x) = \sum_{j=1}^{N} f_j(x) e^{-\varepsilon(h_{j+1}(x)+\ldots+h_N(x))}.$$

Lemma (2)

Let α be the constant in lemma 1. For $1 \leqslant k \leqslant N$,

$$\left| \int_{-\frac{T}{2}}^{\frac{T}{2}} (L_0 - L_1)(x) e^{2i\pi\lambda_k x} \varphi(x) \, \mathrm{d}x \right| \leqslant \frac{\frac{2\alpha}{3}}{k + N_T} \tag{2}$$

Proof

Lemma (1) + lemma (2) implies the theorem with

$$\sum_{k=1}^{N} \frac{|a_k|}{k+N_T} \leqslant \frac{3}{\alpha} \|L_1\|_\infty \|\varphi\|_\infty \|\phi\|_1,$$

Proof

thus we get

$$\sum_{k=1}^{N}\frac{|a_k|}{k}\leqslant(1+N_T)\sum_{k=1}^{N}\frac{|a_k|}{k+N_T}\leqslant\frac{3}{\alpha}(1+N_T)\|L_1\|_\infty\|\varphi\|_\infty\|\phi\|_1.$$

We multiply both inequalities (1) and (2) by $|a_k|$, using the triangle inequality and summing over k, we get

$$\left|\sum_{k=1}^{N}\frac{|a_k|}{k+N_T}-\int_{-\frac{T}{2}}^{\frac{T}{2}}L_0(x)\phi(x)\varphi(x)\,\mathrm{d}x\right|\leqslant(1-\alpha)\sum_{k=1}^{N}\frac{|a_k|}{k+N_T}.$$

$$\left|\int_{-\frac{T}{2}}^{\frac{T}{2}}(L_0-L_1)(x)\phi(x)\varphi(x)\,\mathrm{d}x\right|\leqslant\frac{2\alpha}{3}\sum_{k=1}^{N}\frac{|a_k|}{k+N_T}$$

Adding these two inequalities, we get

$$\frac{\alpha}{3}\sum_{k=1}^{N}\frac{|a_k|}{k+N_T} \leqslant \left|\int_{-\frac{T}{2}}^{\frac{T}{2}} L_1(x)\phi(x)\varphi(x)\,\mathrm{d}x\right| \leqslant \|L_1\|_\infty\|\varphi\|_\infty\|\phi\|_1$$

And finally,

$$\sum_{k=1}^{N} \frac{|a_k|}{k + N_T} \leqslant B_T \|\phi\|_1$$

where $B_T = \dfrac{3}{\alpha}\|L_1\|_\infty . \|\varphi\|_\infty$.

Quadratic frequences : $\lambda_k = k^2$

Theorem (Zalcwasser 1936)

There exists $C > 0$ such that

$$C\sqrt{N} \leqslant \int_{-1}^{1} \left| \sum_{k=0}^{N} e^{i\pi k^2 x} \right| \mathrm{d}x,$$

Quadratic frequences : $\lambda_k = k^2$

we would like to generalise this outcome to trigonometric polynomial of the following form

$$\sum_{k=0}^{N} a_k e^{i\pi k^2 x},$$

for any sequence $(a_k)_{1 \leqslant k \leqslant N}$ of complex numbers.

Result

Theorem

There exists $C > 0$, such that for any sequence a_k of complex number, we have

$$\int_{-1}^{1} \left| \sum_{k=0}^{N} a_k e^{i\pi k^2 x} \right| \mathrm{d}x \geqslant C \min_{0 \leqslant k \leqslant N} |a_k|^{\frac{1}{1-\theta}} \left(|a_0| + \sum_{k \geqslant 1} |a_k - a_{k-1}| \right)^{\frac{-\theta}{1-\theta}} \sqrt{N}$$

where $\theta \in \left] \frac{1}{2}, 1 \right[$.

Result

Theorem

There exists $C > 0$, such that for any sequence a_k of complex number, we have

$$\int_{-1}^{1} \left| \sum_{k=0}^{N} a_k e^{i\pi k^2 x} \right| \mathrm{d}x \geqslant C \min_{0 \leqslant k \leqslant N} |a_k|^{\frac{1}{1-\theta}} \left(|a_0| + \sum_{k \geqslant 1} |a_k - a_{k-1}| \right)^{\frac{-\theta}{1-\theta}} \sqrt{N}$$

where $\theta \in \left] \frac{1}{2}, 1 \right[$.

Key words of the proof :

Result

Theorem

There exists $C > 0$, such that for any sequence a_k of complex number, we have

$$\int_{-1}^{1} \left| \sum_{k=0}^{N} a_k e^{i\pi k^2 x} \right| \mathrm{d}x \geqslant C \min_{0\leqslant k\leqslant N} |a_k|^{\frac{1}{1-\theta}} \left(|a_0| + \sum_{k\geqslant 1} |a_k - a_{k-1}| \right)^{\frac{-\theta}{1-\theta}} \sqrt{N}$$

where $\theta \in \left]\frac{1}{2}, 1\right[$.

Key words of the proof :

- Residue theorem

Result

Theorem

There exists $C > 0$, such that for any sequence a_k of complex number, we have

$$\int_{-1}^{1} \left| \sum_{k=0}^{N} a_k e^{i\pi k^2 x} \right| \mathrm{d}x \geqslant C \min_{0 \leqslant k \leqslant N} |a_k|^{\frac{1}{1-\theta}} \left(|a_0| + \sum_{k \geqslant 1} |a_k - a_{k-1}| \right)^{\frac{-\theta}{1-\theta}} \sqrt{N}$$

where $\theta \in \left] \frac{1}{2}, 1 \right[$.

Key words of the proof :

- Residue theorem
- Continued fraction

Result

Theorem

There exists $C > 0$, such that for any sequence a_k of complex number, we have

$$\int_{-1}^{1} \left| \sum_{k=0}^{N} a_k e^{i\pi k^2 x} \right| \mathrm{d}x \geqslant C \min_{0 \leqslant k \leqslant N} |a_k|^{\frac{1}{1-\theta}} \left(|a_0| + \sum_{k \geqslant 1} |a_k - a_{k-1}| \right)^{\frac{-\theta}{1-\theta}} \sqrt{N}$$

where $\theta \in \left] \frac{1}{2}, 1 \right[$.

Key words of the proof :

- Residue theorem
- Continued fraction
- Layer Cake representation

Result

Theorem

There exists $C > 0$, such that for any sequence a_k of complex number, we have

$$\int_{-1}^{1} \left| \sum_{k=0}^{N} a_k e^{i\pi k^2 x} \right| \mathrm{d}x \geqslant C \min_{0 \leqslant k \leqslant N} |a_k|^{\frac{1}{1-\theta}} \left(|a_0| + \sum_{k \geqslant 1} |a_k - a_{k-1}| \right)^{\frac{-\theta}{1-\theta}} \sqrt{N}$$

where $\theta \in \left] \frac{1}{2}, 1 \right[$.

Key words of the proof :

- Residue theorem
- Continued fraction
- Layer Cake representation
- Interpolation

References

Hardy, G. H. ; Littlewood, J. E. A new proof of a theorem on rearrangements. J. London Math. Soc. 23 (1948), 163–168.

McGehee, O. Carruth ; Pigno, Louis ; Smith, Brent, Hardy's inequality and the L1 norm of exponential sums. Ann. of Math. (2) 113 (1981), no. 3, 613-618.

Choimet, D. ; Queffélec, H. Analyse mathématique : grands théorème du vingtième siècle. Calvage & Mounet 2009.

Nazarov, F. L. On a proof of the Littlewood conjecture by McGehee, Pigno and Smith. St. Petersburg Math. J. 7 (1996), no. 2, 265-275

Zalcwasser, Z. Sur les polynomes associés aux fonctions modulaires. Studia Mathematica. 7 (1938), no 1, 16-35 ϑ.

Mordell, L.J. The approximate functional formula for the theta function. Journal of the London Mathematical Society, 1(2) :68-72, 1926

Thank you