From Ingham's to Nazarov's inequality

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Exponential Sums

A non-harmonic trigonometric polynomial is an expression of the form

$$\sum_{k=0}^{N} a_k e^{2i\pi\lambda_k t}.$$

We are interested in lower bounds of the L^1-norm or the Besicovitch $\mathcal{B}^1$-norm of such trigonometric polynomials, where, for $1 \leq p < +\infty$, the Besicovitch $\mathcal{B}_p$-norms are defined by

$$\|\Phi\|_{\mathcal{B}_p}^p = \lim_{T\to+\infty} \frac{1}{T} \int_{[-T/2,T/2]} |\Phi(x)|^p \, \mathrm{d}x.$$

Those norms can be seen as a substitute to $L^p([-1/2,1/2])$-norms to investigate non-harmonic trigonometric polynomials.

Littlewood Conjecture

In 1948, Littlewood tempted to ask the following question: Is it true that, when $n_0 < n_1 < \cdots < n_N$ are integers

$$\int_{-1/2}^{1/2} \left| \sum_{k=0}^{N} e^{2i\pi n_k t} \right| \mathrm{d}t \geq \int_{-1/2}^{1/2} \left| \sum_{k=0}^{N} e^{2i\pi kt} \right| \mathrm{d}t.$$

Littlewood did not explicitly ask this question and made a safer guess. He conjectured that, when $n_0 < n_1 < \cdots < n_N$ are integers, there exists a universal constant C such that

$$L_N := \inf_{n_0 < n_1 < \cdots < n_N} \int_{-1/2}^{1/2} \left| \sum_{k=0}^{N} e^{2i\pi n_k t} \right| \mathrm{d}t \geq C \ln(N+1).$$

for some constant

$$C \leq \frac{4}{\pi^2}.$$

Partial results

The first non-trivial estimate was obtained by Cohen who proved that

$$L_N \geq C(\ln N/\ln\ln N)^{1/8}.$$

Subsequent improvements are due to Pichorides who proved that

$$L_N \geq C\ln N/(\ln\ln N)^2.$$

Solutions to the conjecture

Theorem (McGehee, Pigno & Smith 1981)

For $n_0 < n_1 < \cdots < n_N$ integers and $a_0, \ldots, a_N$ complex numbers,

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \sum_{k=0}^{N} a_k e^{2i\pi n_k t} \right| \mathrm{d}t \geq \frac{1}{30} \sum_{k=0}^{N} \frac{|a_k|}{k+1}$$

Solutions to the conjecture

Theorem (Stegeman & Yabuta 1982)

For $n_0 < n_1 < \cdots < n_N$ integers and $a_0, \ldots, a_N$ complex numbers all of modulus larger than 1

$$\int_{-1/2}^{1/2} \left| \sum_{k=0}^{N} a_k e^{2i\pi n_k t} \right| \mathrm{d}t \geq \frac{4}{\pi^3} \ln(N+1).$$

Quadratic frequencies

Let $a = (a_k)_{k=0,\dots,N}$ be a sequence of complex numbers, we write

$$\mathbb{E}\left[|a|\right] = \frac{1}{N+1}\sum_{k=0}^{N}|a_k| \quad \text{and} \quad \|\partial a\|_{1,N} = |a_0| + \sum_{k=1}^{N}|a_k - a_{k-1}|,$$

Thus for the constant sequence $a = 1$, Zalcwasser proved the following

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\sum_{k=0}^{N} e^{2i\pi k^2 x}\right| \mathrm{d}x \geq C\sqrt{N}\left(\mathbb{E}[|a|^2]\right)^{1/2}, \tag{1}$$

for some $C \geq 0$. Our first result is an extension of (1)

Theorem (Jaming, Kellay & Saba)

For every $\varepsilon > 0$ there exists a constant C_ε such that if $a = (a_k)_{k=0,\dots,N}$ is a sequence of complex numbers, then

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\sum_{k=0}^{N} a_k e^{2i\pi k^2 x}\right| \mathrm{d}x \geq C_\varepsilon\sqrt{N}\left(\frac{\left(\mathbb{E}[|a|^2]\right)^{\frac{1}{2}}}{\|\partial a\|_{1,N}}\right)^{2+\varepsilon}\left(\mathbb{E}[|a|^2]\right)^{\frac{1}{2}}. \tag{2}$$

First Results in Real Frequencies Setting

Theorem (Hudson & Leckband 1992)

For $\lambda_0 < \lambda_1 < \ldots < \lambda_N$ real numbers and $a_0, \ldots, a_N$ complex numbers,

$$\lim_{T \to +\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left| \sum_{k=0}^{N} a_k e^{2i\pi\lambda_k t} \right| \mathrm{d}t \geq \frac{1}{30} \sum_{k=0}^{N} \frac{|a_k|}{k+1}.$$

Later, Nazarov showed that such a result holds as soon as $T > 1$:

Theorem (Nazarov 1996)

For $T > 1$, there exists a constant C_T such that, for $0 < \lambda_0 < \cdots < \lambda_N$ real numbers verifying $|\lambda_k - \lambda_\ell| \geq |k - \ell|$ and $a_0, \ldots, a_N$ complex numbers,

$$\int_{-T/2}^{T/2} \left| \sum_{k=0}^{N} a_k e^{2i\pi\lambda_k t} \right| \mathrm{d}t \geq C_T \sum_{k=0}^{N} \frac{|a_k|}{k+1}. \tag{3}$$

Theorem (Jaming, Kellay & Saba 2023/2024)

Let $\lambda_0 < \lambda_1 < \cdots < \lambda_N$ *be distinct real numbers and* $a_0, \ldots, a_N$ *be complex numbers. Then*

1. *we have*

$$\lim_{T\to+\infty} \frac{1}{T}\int_{-T/2}^{T/2} \left|\sum_{k=0}^{N} a_k e^{2i\pi\lambda_k t}\right| \mathrm{d}t \geq \frac{1}{26}\sum_{k=0}^{N}\frac{|a_k|}{k+1}.$$

2. *If further* $a_0, \ldots, a_N$ *all have modulus larger than* 1, $|a_k| \geq 1$ *then*

$$\lim_{T\to+\infty} \frac{1}{T}\int_{-T/2}^{T/2} \left|\sum_{k=0}^{N} a_k e^{2i\pi\lambda_k t}\right| \mathrm{d}t \geq \frac{4}{\pi^3}\ln(N+1).$$

3. *Assume further that for* $k = 0, \ldots, N-1$, $\lambda_{k+1} - \lambda_k \geq 1$, *then, for every* $T > 1$, *there exists a constant* $C(T)$ *such that, for every* $a_0, \ldots, a_N \in \mathbb{C}$,

$$\frac{1}{T}\int_{-T/2}^{T/2} \left|\sum_{k=0}^{N} a_k e^{2i\pi\lambda_k t}\right| \mathrm{d}t \geq C(T)\sum_{k=0}^{N}\frac{|a_k|}{k+1}.$$

$$\frac{1}{T}\int_{-T/2}^{T/2}\left|\sum_{k=0}^{N} a_k e^{2i\pi\lambda_k t}\right| \mathrm{d}t \geq C(T)\sum_{k=0}^{N}\frac{|a_k|}{k+1}.$$

Moreover,

1. for $T \geq 72$ we can take $C(T) = \frac{1}{122}$;
2. for $1 < T \leq 2$, $C(T) = O\big((T-1)^{15/2}\big)$.
3. For $2 < T < 72$, (10) follows from the case (b) with $T = 2$.

L^2-setting

Theorem (Ingham 1936)

Let $\gamma > 0$ *and* $T > \dfrac{1}{\gamma}$. *Then there exist* $0 < A_2(T,\gamma) \leq B_2(T,\gamma)$ *such that*

– for every sequence of real numbers $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ *such that* $\lambda_{k+1} - \lambda_k \geq \gamma$;

– for every sequence $(a_k)_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z},\mathbb{C})$,

$$A_2(T,\gamma)\sum_{k\in\mathbb{Z}}|a_k|^2 \leq \frac{1}{T}\int_{-T/2}^{T/2}\left|\sum_{k\in\mathbb{Z}} a_k e^{2i\pi\lambda_k t}\right|^2 dt \leq B_2(T,\gamma)\sum_{k\in\mathbb{Z}}|a_k|^2$$

L^2-setting

Theorem (Kahane 1962)

Let $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ *such that* $\lambda_{k+1} - \lambda_k \to +\infty$ *when* $k \to \pm\infty$. *Then, for every* $T > 0$, *there exist constants* $0 < A_2(T,\Lambda) \leq B_2(T,\Lambda)$ *such that*

$$A_2(T,\Lambda)\sum_{k\in\mathbb{Z}}|a_k|^2 \leq \frac{1}{T}\int_{-T/2}^{T/2}\left|\sum_{k\in\mathbb{Z}} a_k e^{2i\pi\lambda_k t}\right|^2 dt \leq B_2(T,\Lambda)\sum_{k\in\mathbb{Z}}|a_k|^2$$

holds for every sequence $(a_k)_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z},\mathbb{C})$.

L^1-analogue of Kahane's Theorem

Theorem (Jaming, Kellay, Saba & Wang 2024)

Let $\Lambda = (\lambda_k)_{k\in\mathbb{Z}}$ be an increasing sequence with $\lambda_{k+1} - \lambda_k \to +\infty$ when $k \to \pm\infty$. Then, for every $T > 0$, there exists a constant $\tilde{A}_1(T,\Lambda) > 0$ such that, if $(a_k)_{k\in\mathbb{N}} \subset \mathbb{C}$ is a sequence of complex numbers, and $N \geq 1$, then

$$\tilde{A}_1(T,\Lambda)\sum_{k=0}^{N}\frac{|a_k|}{1+k} \leq \frac{1}{T}\int_{-T/2}^{T/2}\left|\sum_{k=0}^{N} a_k e^{2i\pi\lambda_k t}\right|\,\mathrm{d}t.$$

If further $\displaystyle\sum_{k\in\mathbb{Z}}\frac{1}{1+|\lambda_k|}$ converges, then there also exists a constant $A_1(T,\Lambda)$ such that, for every $(a_k)_{k\in\mathbb{Z}} \subset \mathbb{C}$ and every $N \geq 1$,

$$A_1(T,\Lambda)\max_{k=-N,\dots,N}|a_k| \leq \frac{1}{T}\int_{-T/2}^{T/2}\left|\sum_{k=-N}^{N} a_k e^{2i\pi\lambda_k t}\right|\,\mathrm{d}t.$$

Perspectives

1. Validity of Nazarov's theorem in the case $T = 1$;
2. Generalization to sets with multidimensional structure;

References

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Thank you